Euler characteristics of $\mathrm{Out}(F_n)$ and renormalized topological field theory

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joint work with Karen Vogtmann

`arXiv:1907.03543`

`arXiv:2202.08739`

`arXiv:22xx`

Euler characteristics

- Usual (classical) Euler characteristic for a space X:

$$\widetilde{\chi}(X) = \sum_k (-1)^k \dim H_k(X, \mathbb{Q})$$

- Virtual/orbifold Euler characteristic with group G acting on X:

$$\chi(X/G) = \sum_{\langle \sigma \rangle} \frac{(-1)^{\dim \sigma}}{|\operatorname{Stab}_G \sigma|}$$

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cells of a subdivision of X

orbit representatives under G-action

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$$\chi(X/G) = \chi(G)$$

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- If G is virtually torsion-free, acts properly and cocompactly, and X is contractible, then $\chi(X/G)$ is an invariant of G.
- This invariant behaves well under morphisms

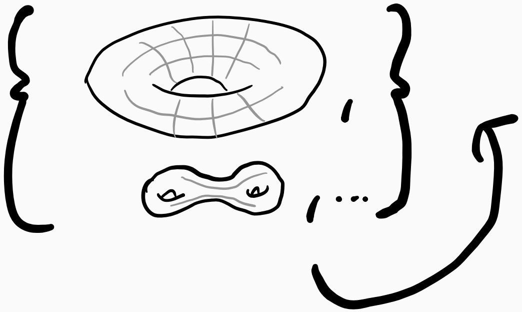
$$1 \to G \to H \to M \to 1 \qquad \Rightarrow \qquad \chi(H) = \chi(G) \cdot \chi(M)$$

the classic Euler characteristic does not

$$\not\Rightarrow \qquad \widetilde{\chi}(H) = \widetilde{\chi}(G) \cdot \widetilde{\chi}(M)$$

Overview

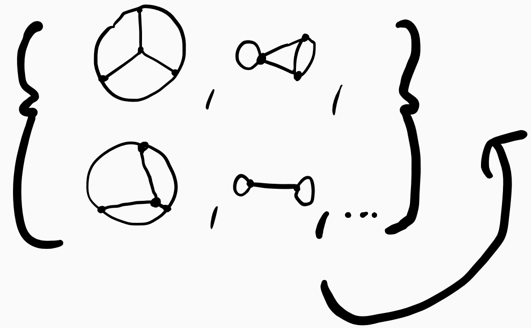
Teichmüller space $\mathbb{T}_g$



$\mathrm{MCG}(S_g)$

mapping class group

Culler-Vogtmann Outer space $\mathcal{O}_n$



$\mathrm{Out}(F_n)$

outer automorphisms of F_n

Harer Zagier (1986):

$$\chi(\mathrm{MCG}(S_g)) = \chi(\mathcal{M}_g) = \frac{B_{2g}}{4g(g-1)}$$

Here:

$$\chi(\mathrm{Out}(F_n)) = \chi(\mathcal{O}_n \,/\, \mathrm{Out}(F_n)) = \ldots < 0$$

Euler characteristics

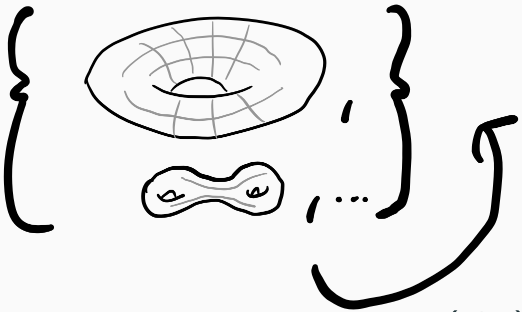
- Usual (classical) Euler characteristic for a space X:

$$\widetilde{\chi}(X) = \sum_k (-1)^k \dim H_k(X, \mathbb{Q})$$

$$\Rightarrow \dim H_\bullet(X, \mathbb{Q}) \geq |\tilde{\chi}(X)|$$

Overview

Teichmüller space $\mathbb{T}_g$



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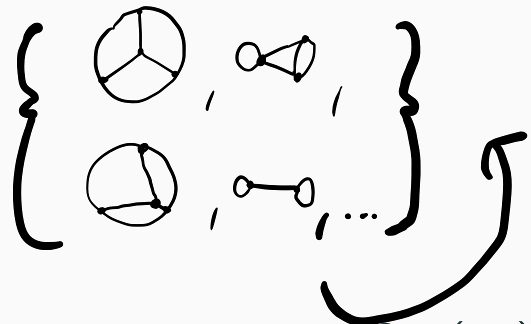

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Harer Zagier (1986):

$$\chi(\mathrm{MCG}(S_g)) = \chi(\mathcal{M}_g)$$

$$= \frac{B_{2g}}{4g(g-1)}$$

$$|\widetilde{\chi}(\mathrm{MCG}(S_g))| \sim g^{2g}$$

Here:

$$\chi(\mathrm{Out}(F_n)) = \chi(\mathcal{O}_n \,/\, \mathrm{Out}(F_n))$$

$$= \ldots < 0$$

$$|\widetilde{\chi}(\mathrm{Out}(F_n))| \sim n^n$$

Groups

Automorphisms of groups

- Take a group G
- An automorphism of G, $\rho \in \mathrm{Aut}(G)$ is a bijection

$$\rho : G \to G$$

$$\text{such that } \rho(x \cdot y) = \rho(x) \cdot \rho(y) \text{ for all } x, y \in G$$

- Normal subgroup: $\mathrm{Inn}(G) \triangleleft \mathrm{Aut}(G)$, the inner automorphisms.
- We have, $\rho_h \in \mathrm{Inn}(G)$

$$\rho_h : G \to G,$$

$$g \mapsto h^{-1}gh$$

  for each $h \in G$.
- Outer automorphisms: $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$

Automorphisms of the free group

- Consider the free group with n generators

$$F_n = \langle a_1, \ldots, a_n \rangle$$

E.g. $a_1 a_3^{-5} a_2 \in F_3$ only identity: $a_k a_k^{-1} = \mathrm{id}$

- The group $\mathrm{Out}(F_n)$ is our main object of interest.
- Generated by

$$\begin{array}{lllll} & a_1 \mapsto a_1 a_2 & a_2 \mapsto a_2 & a_3 \mapsto a_3 & \ldots \\ \text{and} & a_1 \mapsto a_1^{-1} & a_2 \mapsto a_2 & a_3 \mapsto a_3 & \ldots \end{array}$$

and permutations of the letters.

Spaces

How to study such groups?

How to study groups such as $\mathrm{MCG}(S)$ or $\mathrm{Out}(F_n)$?

Main idea

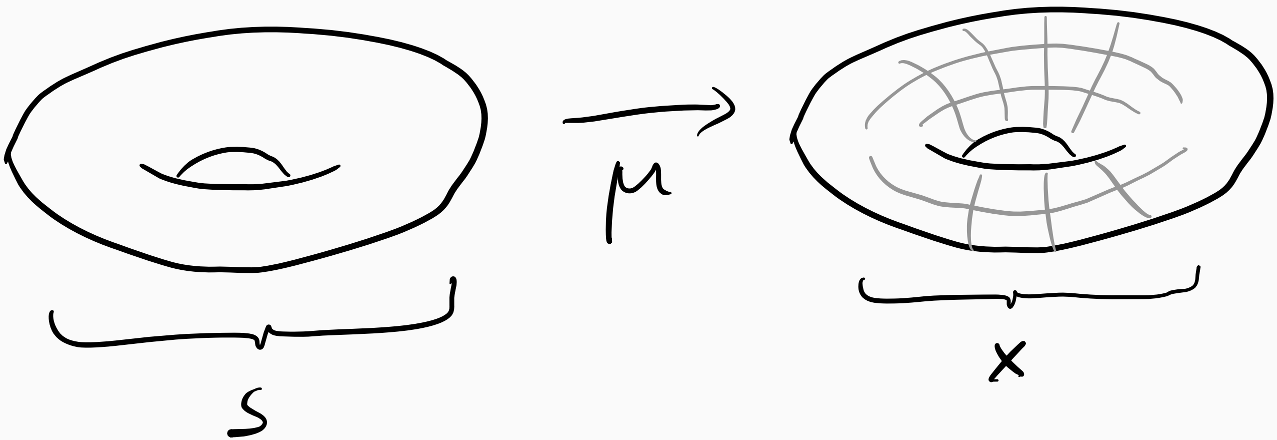
Realize G as symmetries of some geometric object.

For the mapping class group: Teichmüller space

Let S be a closed, connected and orientable surface.

$\Rightarrow$ A point in Teichmüller space $T(S)$ is a pair, (X, μ)

- A Riemann surface X.
- A *marking*: a homeomorphism $\mu : S \to X$.



MCG(S) acts on $T(S)$ by composing to the marking:

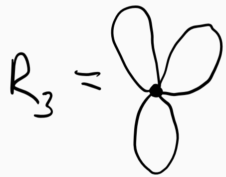
$$(X, \mu) \mapsto (X, \mu \circ g^{-1}) \text{ for some } g \in \mathrm{MCG}(S).$$

For $\mathrm{Out}(F_n)$: Outer space

Idea: Mimic previous construction for $\mathrm{Out}(F_n)$.

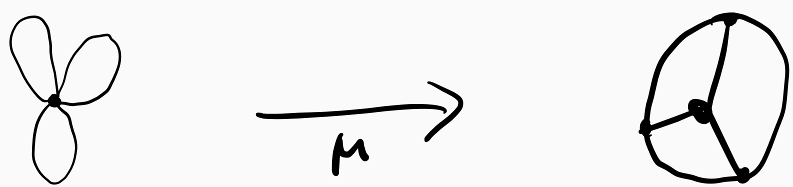
Culler, Vogtmann (1986)

Let R_n be the rose with n petals.



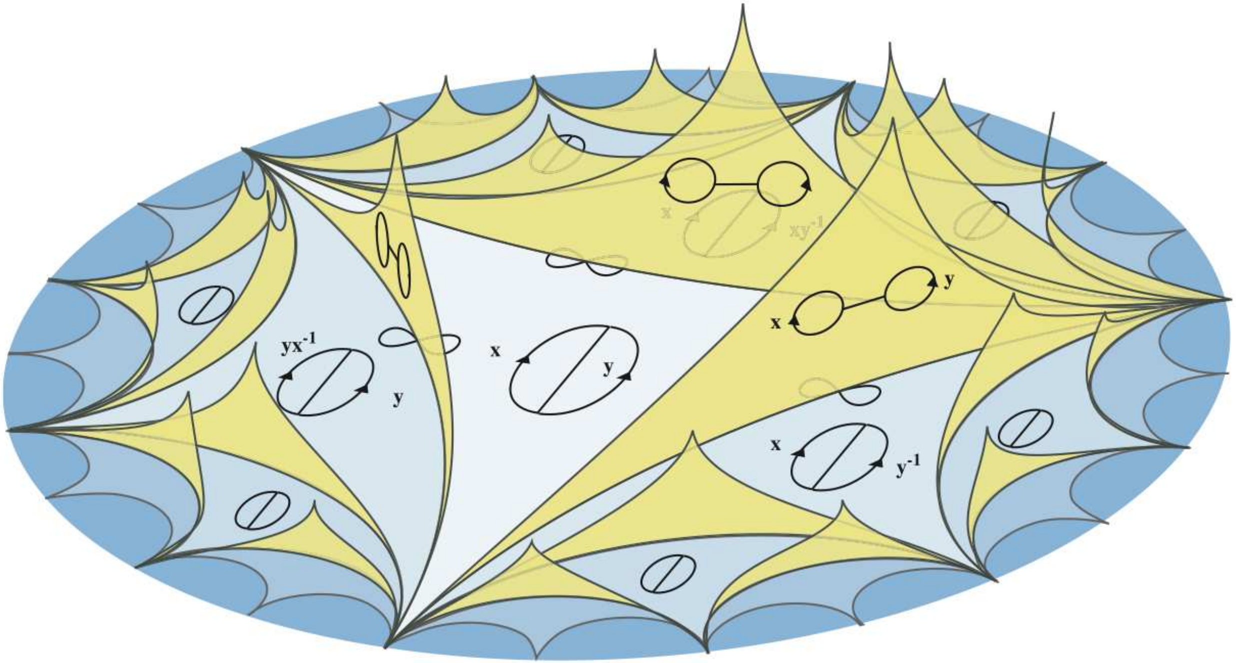
$\Rightarrow$ A point in Outer space $\mathcal{O}_n$ is a pair, (G, μ)

- A connected graph G with a length assigned to each edge.
- A marking: a *homotopy equivalence* $\mu : R_n \to G$.



$\mathrm{Out}(F_n)$ acts on $\mathcal{O}_n$ by composing to the marking:

$$(\Gamma, \mu) \mapsto (\Gamma, \mu \circ g^{-1}) \text{ for some } g \in \mathrm{Out}(F_n) = \mathrm{Out}(\pi_1(R_n)).$$



Vogtmann 2008

Intermezzo: Brief physics perspective

Roughly:

$$\text{Scalar QFT} \sim \text{Integrals over } \mathcal{O}_n \,/\, \mathrm{Out}(F_n)$$

analogous to

$$\text{2D Quantum gravity} \sim \text{Integral over } T(S)/\,\mathrm{MCG}(S)$$

Summary of the respective groups and spaces

	$\mathrm{MCG}(S_g)$	$\mathrm{Out}(F_n)$
acts freely and properly on	Teichmüller space $\mathcal{T}(S_g)$	Outer space $\mathcal{O}_n$
Quotient X/G	Moduli space of curves $\mathcal{M}_g$	Moduli space of graphs $\mathcal{O}_n \,/\, \mathrm{Out}(F_n)$

Invariants

Why study $\chi(\mathrm{Out}\, F_n)$?

Further motivation to look at Euler characteristic of $\mathrm{Out}(F_n)$

Consider the abelization map $F_n \to \mathbb{Z}^n$.

$\Rightarrow$ Induces a group homomorphism

$$1 \to \mathcal{T}_n \to \mathrm{Out}(F_n) \to \underbrace{\mathrm{Out}(\mathbb{Z}^n)}_{=\mathrm{GL}(n,\mathbb{Z})} \to 1$$

- $\mathcal{T}_n$ the 'non-abelian' part of $\mathrm{Out}(F_n)$ is interesting.
- By the short exact sequence above

$$\chi(\mathrm{Out}(F_n)) = \underbrace{\chi(\mathrm{GL}(n,\mathbb{Z}))}_{=0}\chi(\mathcal{T}_n) \quad n \geq 3$$

$\Rightarrow$ $\chi(\mathrm{Out}(F_n)) = 0$ for $n \geq 3$?

Further motivation to look at Euler characteristic of $\mathrm{Out}(F_n)$

But :

n	2	3	4	5	6	...
$\chi(\mathrm{Out}\,F_n)$	$-\frac{1}{24}$	$-\frac{1}{48}$	$-\frac{161}{5760}$	$-\frac{367}{5760}$	$-\frac{120257}{580608}$	

(Smillie Vogtmann 87)

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$\Rightarrow$ $\chi(\mathrm{Out}(F_n)) = 0$ for $n \geq 3$? No!

$\Rightarrow$ $\mathcal{T}_n$ does not have finitely-generated homology for $n \geq 3$ if $\chi(\mathrm{Out}(F_n)) \neq 0$.

Conjectures

Conjecture Smillie-Vogtmann (1987)

$$\chi(\mathrm{Out}(F_n)) \neq 0 \text{ for all } n \geq 2$$

$$\text{and } |\chi(\mathrm{Out}(F_n))| \text{ grows exponentially for } n \to \infty.$$

based on initial computations by Smillie-Vogtmann (1987) up to $n \leq 11$. Later strengthened by Zagier (1989) up to $n \leq 100$.

Conjecture Magnus (1934)

$\mathcal{T}_n$ is not finitely presentable.

In topological terms, i.e. $\dim(H_2(\mathcal{T}_n)) = \infty$,

which implies that $\mathcal{T}_n$ does not have finitely-generated homology.

Theorem Bestvina, Bux, Margalit (2007)

$\mathcal{T}_n$ does not have finitely-generated homology.

Result: $\chi(\mathrm{Out}(F_n)) \neq 0$

Theorem A MB-Vogtmann (2019)

$$\chi(\mathrm{Out}(F_n)) < 0 \text{ for all } n \geq 2$$

$$\chi(\mathrm{Out}(F_n)) \sim -\frac{1}{\sqrt{2\pi}} \frac{\Gamma(n-3/2)}{\log^2 n} \text{ as } n \to \infty.$$

which settles the initial conjecture by Smillie-Vogtmann (1987).

This Theorem A follows from an implicit expression for $\chi(\mathrm{Out}(F_n))$:

Theorem B MB-Vogtmann (2019)

$$\sqrt{2\pi}e^{-N}N^N \sim \sum_{k\geq 0} a_k(-1)^k\Gamma(N+1/2-k) \text{ as } N\to\infty$$

$$\text{where } \sum_{k\geq 0} a_k z^k = \exp\left(\sum_{n\geq 1}\chi(\mathrm{Out}(F_{n+1}))z^n\right)$$

$\Rightarrow$ $\chi(\mathrm{Out}(F_n))$ are the coefficients of an asymptotic expansion.

- Analytic argument needed to prove Theorem B $\Rightarrow$ Theorem A.
- In this talk: Focus on proof of Theorem B

Analogy to the mapping class group

Harer-Zagier formula for $\chi(\mathrm{MCG}(S_g))$

Similar result for the mapping class group/moduli space of curves:

Theorem Harer-Zagier (1986)

$$\chi(\mathcal{M}_g) = \chi(\mathrm{MCG}(S_g)) = \frac{B_{2g}}{4g(g-1)} \qquad g \geq 2$$

- Original proof by Harer and Zagier in 1986.
- Alternative proof using topological field theory (TFT) by Penner (1988).
- Simplified proof by Kontsevich (1992) based on TFT's.

$\Rightarrow$ Kontsevich's proof served as a blueprint for $\chi(\mathrm{Out}(F_n))$.

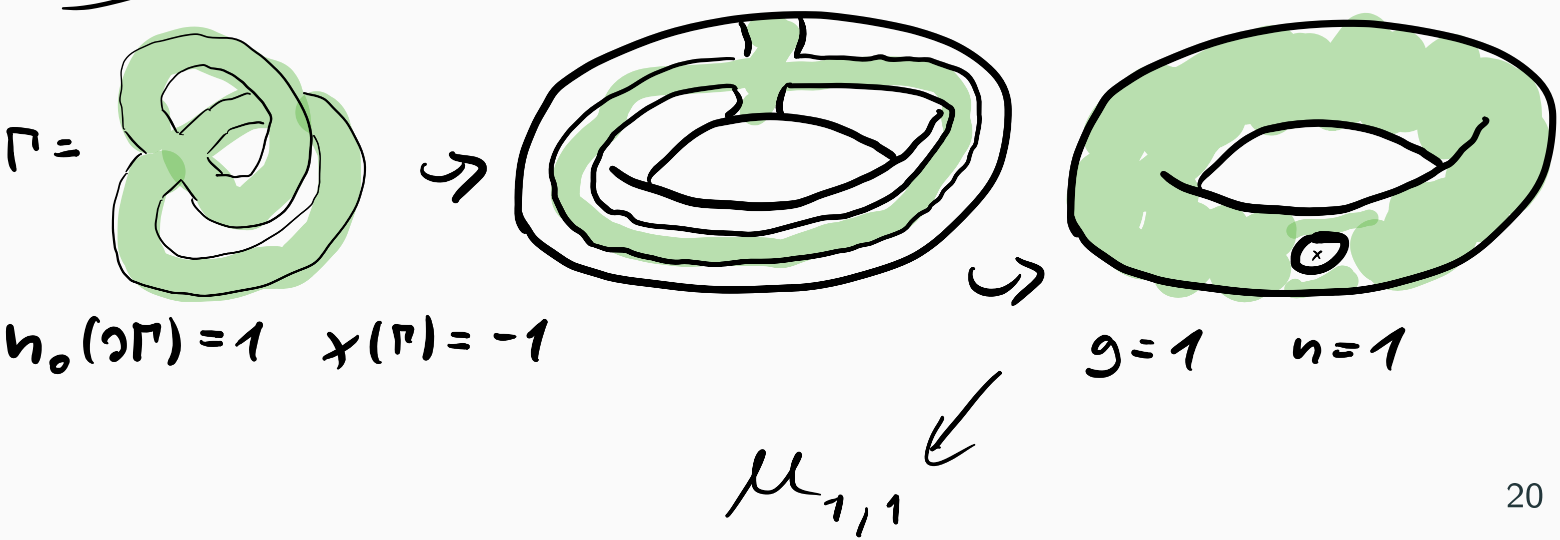
Kontsevich's argument

- We have the identity by Kontsevich (1992):

$$\sum_{g,n} \frac{\chi(\mathcal{M}_{g,n})}{n!} z^{2-2g-n} = \sum_{\text{connected graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)}.$$

- Kontsevich proved this using a combinatorial model of $\mathcal{M}_{g,n}$ by Penner (1986) based on ribbon graphs.

For instance:



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- The expression on the right hand side can be evaluated using a 'topological field theory':

$$\sum_{\text{connected graphs G}} \frac{(-1)^{|V_G|}}{|\operatorname{Aut} G|} z^{\chi(G)} = \log\left(\frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)} dx\right)$$

$$= \sum_{k\geq 1} \frac{\zeta(-k)}{-k} z^{-k}$$

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TFT action

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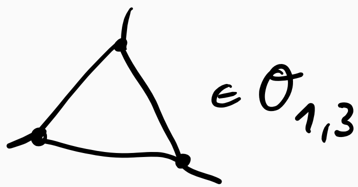
- The formula for $\chi(\mathcal{M}_{g,n})$ follows via the short exact sequence

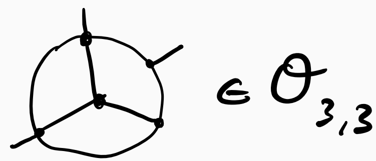
$$1 \to \pi_1(S_{g,n}) \to \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n} \to 1$$

Analogous proof strategy for $\chi(\mathrm{Out}(F_n))$ using renormalized TFTs

Step 1

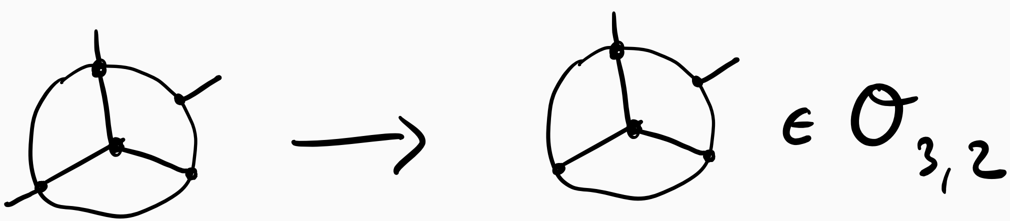
Generalize from $\mathrm{Out}(F_n)$ to $A_{n,s}$ and from $\mathcal{O}_n$ to $\mathcal{O}_{n,s}$, Outer space of graphs of rank n and s legs.
Contant, Kassabov, Vogtmann (2011)





Forgetting a leg gives the short exact sequence of groups

$$1 \to F_n \to A_{n,s} \to A_{n,s-1} \to 1$$

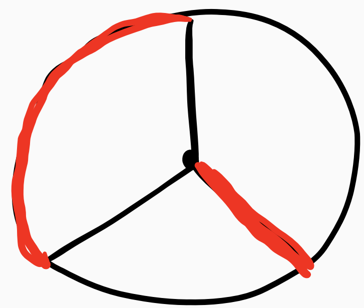


Step 2

- Use a combinatorial model for $\mathcal{G}_{n,s}$

$\Rightarrow$ graphs with a forest Smillie-Vogtmann (1987):

A ~~point~~ cell in $\mathcal{G}_{n,s}$ can be associated with a pair of a graph G and a forest $f \subset G$.



$f \subset G$

(G, f)

graph ↗ ↖ forest

Step 3

$$\chi(A_{n,s}) = \sum_{\sigma} \frac{(-1)^{\dim(\sigma)}}{|\mathrm{Stab}(\sigma)|}$$

dimension of resp. strata $= |E_f|$

stabilizer under action of $A_{n,s}$ $= |\mathrm{Aut}(G,f)|$

sum over representatives of cells of $\mathcal{O}_{n,s}/A_{n,s}$

$\rightsquigarrow$ legged graphs G with forest f

Step 3

$$\chi(A_{n,s}) = \sum_{\sigma} \frac{(-1)^{\dim(\sigma)}}{|\operatorname{Stab}(\sigma)|}$$

$$= \sum_{\substack{\text{graphs } G \\ \text{with } s \text{ legs} \\ \operatorname{rank}(\pi_1(G))=n}} \sum_{\text{forests } f \subset G} \frac{(-1)^{|E_f|}}{|\operatorname{Aut} G|}$$

Step 4

Renormalized TFT interpretation MB-Vogtmann (2019):

$$\chi(A_{n,s}) = \sum_{\substack{\text{graphs } G \\ \text{with } s \text{ legs} \\ \text{rank}(\pi_1(G))=n}} \frac{1}{|\text{Aut } G|} \underbrace{\sum_{\text{forests } f \subset G} (-1)^{|E_f|}}_{=:\tau(G)}$$

τ fulfills the identities $\tau(\emptyset) = 1$ and

$$\sum_{\substack{g \subset G \\ g \text{ bridgeless}}} \tau(g)(-1)^{|E_{G/g}|} = 0 \qquad \text{for all } G \neq \emptyset$$

$\Rightarrow$ τ is an inverse of a character in a Connes-Kreimer-type renormalization Hopf algebra. Connes-Kreimer (2001)

TFT evaluation

Let $$T(z,x) = \sum_{n,s\geq 0} \chi(A_{n,s}) z^{1-n} \frac{x^s}{s!}$$

then $$1 = \frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{T(z,x)} dx$$

Using the short exact sequence, $1 \to F_n \to A_{n,s} \to A_{n,s-1} \to 1$ results in the action

$$1 = \frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{z(1+x-e^x)+\frac{x}{2}+T(-ze^x)} dx$$

where $T(z) = \sum_{n\geq 1} \chi(\mathrm{Out}(F_{n+1})) z^{-n}$.

This gives the implicit result in Theorem B.

Part 2: The classikal Euler characteristic

Theorem A MB-Vogtmann (2019)

$$\chi(\mathrm{Out}(F_n)) < 0 \text{ for all } n \geq 2$$

$$\chi(\mathrm{Out}(F_n)) \sim -\frac{1}{\sqrt{2\pi}} \frac{\Gamma(n-3/2)}{\log^2 n} \text{ as } n \to \infty.$$

$\Rightarrow$ Indicates huge amount of homology in odd dimensions.

- Where does all this homology come from?

Low rank computations

dim $H_k(\mathrm{Out}(F_n), \mathbb{Q})$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	$\chi(\mathrm{Out}\,F_n)$
2	1	0											1
3	1	0	0	0									1
4	1	0	0	0	1	0							2
5	1	0	0	0	0	0	0	0					1
6	1	0	0	0	0	0	0	0	1	0			2
7	1	0	0	0	0	0	0	0	1	0	0	1	1

1

−21

−124

−1202

Bartholdi 2016

Ohashi 2008

Morita et al 2015

Euler characteristics of Kontsevich's graph complexes

+ Getzler–Kapranov

A missing piece:

complex	virtual χ	classical $\widetilde{\chi}$
associative/$\mathcal{M}_{g,n}$	Harer, Zagier 1986	Harer, Zagier 1986
commutative	Kontsevich 1993	Willwacher, Živković 2015
Lie/Out(F_n)	Kontsevich 1993	?

Lie/Out(F_n) integral case $\widetilde{\chi}(\mathrm{Out}(F_n))$ only known for $n \leq 11$.
Thanks to a supercomputer calculation by Morita 2014.

* Chan, Faber, Galatius, Payne '19

Missing Euler characteristic of the Lie case

Theorem MB, Vogtmann 2022 (in prep)

$$\prod_{n\geq 1}\left(\frac{1}{1-z^{-n}}\right)^{\widetilde{\chi}(\mathrm{Out}(F_{n+1}))} =$$

$$\left(\prod_{k\geq 1}\int \frac{\mathrm{d}\,x_k}{\sqrt{2\pi k/z^k}}\right) e^{\sum_{k\geq 1}\frac{z^k}{k}\left(c_k-\frac{c_{2k}}{2}+\frac{c_k^2}{2}-\frac{x_k^2}{2}-(1+c_k)\sum_{j\geq 1}\frac{\mu(j)}{j}\log(1+c_{jk})\right)}$$

where $c_{2k}=x_{2k}+z^{-k}$ and $c_{2k-1}=x_{2k-1}$ for all $k\geq 1$.

$\Rightarrow$ Getzler-Kapranov type expression for $\widetilde{\chi}(\mathrm{Out}(F_n))$.

(Can be evaluated up to $n\approx 80$ (vs 11 known values).)

using QFT methods!

Theorem MB, Vogtmann 2022 (in prep)

$$\lim_{n\to\infty} \frac{\widetilde{\chi}(\mathrm{Out}\, F_n)}{\chi(\mathrm{Out}\, F_n)} = e^{-\frac{1}{4}}$$

In contrast to $\lim_{g\to\infty} \frac{\widetilde{\chi}(\mathcal{M}_g)}{\chi(\mathcal{M}_g)} = 1$, Harer-Zagier 1986.

Summary and open questions

Short summary:

- $\chi(\mathrm{Out}(F_n)) \neq 0$
- Much unexplained homology of $\mathrm{Out}(F_n)$ due to rapid growth of $\widetilde{\chi}(\mathrm{Out}(F_n))$.

Open questions:

- What generates it?
- The TFT analysis indicates a non-trivial ‘duality’ between $\mathrm{MCG}(S_g)$ and $\mathrm{Out}(F_n)$. Obvious candidate: Koszul duality (?)