Bounds and estimates for Feynman-perturbative expansions

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Renormalized asymptotic enumeration of Feynman diagrams
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Physical motivation

- Often, the perturbation expansions turn out to have **vanishing** radius of convergence!
- Dyson's argument: Let

$$F(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + \dots \quad (1)$$

  be a physical quantity in QED which is calculated as a formal power series in α.
- If F is analytic at $\alpha = 0$ we can analytically continue to negative α, resulting in a QFT **where equal charges attract**.
- The fictitious QFT will have no stable ground state.
  $\Rightarrow$ contradiction $\Rightarrow$ $F(\alpha)$ cannot be analytic at $\alpha = 0$.

First step: Number of diagrams

- The divergence of the perturbative expansion is believed to be caused by the proliferation of Feynman diagrams.
- Feynman diagrams can be counted rather easily using zero-dimensional field theory.
- The integral

$$Z(\hbar) = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2}+F(x)\right)}$$

  is to be interpreted as a formal power series Cvitanović et al. [1978], Argyres et al. [2001], Hurst [1952], Molinari and Manini [2006] .
- Possible 'interactions' are encoded in $F(x)$.

Example

$$Z^{\text{stir}}(\hbar) := \frac{\Gamma\left(\frac{1}{\hbar}\right)}{\sqrt{2\pi\hbar}\left(\frac{1}{\hbar}\right)^{\frac{1}{\hbar}} e^{-\frac{1}{\hbar}}} = \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2}-(e^x-1-x-\frac{x^2}{2})\right)}$$

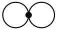
- **Combinatorial integral** representation of Stirling's famous (asymptotic) expansion of the Gamma-function.
- Counts the (orbifold) Euler characteristic of the moduli space of (stable) open curves Kontsevich [1992],

$$\log Z^{\text{stir}}(\hbar) = \sum_{\substack{g,n \\ n+2g-2\geq 0}} \frac{\chi(\mathcal{M}_{g,n})}{n!} \hbar^{n+2g-2}$$

Example

$$Z^{\text{stir}}(\hbar) := \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2} - (e^x - 1 - x - \frac{x^2}{2})\right)}$$

- Set $F(x) = -(e^x - 1 - x - \frac{x^2}{2})$. Combinatorial: All vertices are allowed and $\lambda_k = -1$.
- Diagrammatically:

$Z^{\text{stir}}(\hbar) = 1 + \frac{1}{8}$  $+ \frac{1}{12}$  $+ \frac{1}{8}$  $+ \ldots$

$$= 1 + \hbar \underbrace{\left(\frac{1}{8}(-1)^2 + \frac{1}{12}(-1)^2 + \frac{1}{8}(-1)\right)}_{=\frac{1}{12}} + \ldots$$

$$= 1 + \hbar\frac{1}{12} + \hbar^2\frac{1}{288} - \hbar^3\frac{139}{51840} - \hbar^4\frac{571}{2488320} + \ldots$$

Computation

- Defines a map $\mathcal{F} : x^3\mathbb{R}[[x]] \to \mathbb{R}[[\hbar]]$.
- Suitable for studying **random graphs** Erdös and Rényi [1959].
- Efficient calculation is possible using an interpretation as a hyperelliptic curve.

Interpretation as hyperelliptic curve

$$Z(\hbar) = \sum_{n=0}^{\infty} (2n-1)!![y^{2n}]G'(y)$$

where $G(y)$ is the (positive) solution of $\frac{y^2}{2} = \frac{G(y)^2}{2} - F(G(y))$.

- The implicit equation $\frac{y^2}{2} = \frac{G(y)^2}{2} - F(G(y))$ defines a **complex curve** in $\mathbb{C}^2$.
- The asymptotics of $Z(\hbar)$ are governed by the asymptotics of the **convergent** power series $G(y)$.
- Similar structures to **topological recursion** Eynard and Orantin [2007].

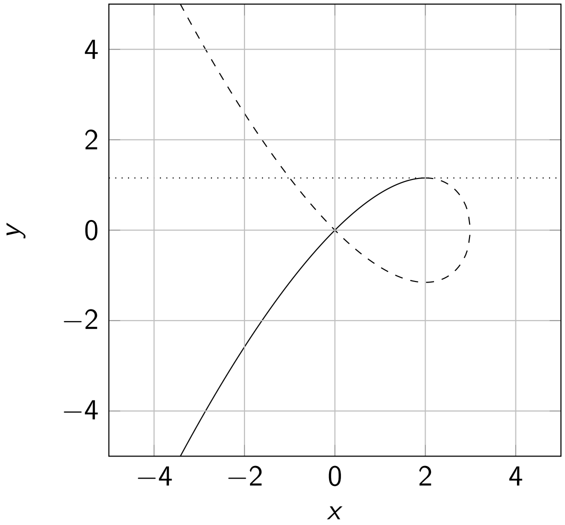


Figure: Plot of the elliptic curve $\frac{y^2}{2} = \frac{x^2}{2} - \frac{x^3}{3!}$, which can be associated to the perturbative expansion of zero-dimensional φ^3-theory. The dominant singularity can be found at $(x, y) = \left(2, \frac{2}{\sqrt{3}}\right)$.

- Renormalization can be used to restrict the number of diagrams.
- Using BPHZ renormalization, the number of skeleton diagrams is obtained.
- More sophisticated techniques can be used to restrict to more general classes of diagrams $\Rightarrow$ Hopf algebra of Feynman diagrams Connes and Kreimer [1999].
- Answers question by Freeman Dyson: Number of skeleton diagrams in quenched QED is

$$e^{-2}(2n-1)!!\left(1-\frac{6}{2n+1}\right.$$
$$\left.-\frac{4}{(2n-1)(2n+1)}-\frac{218}{3}\frac{1}{(2n-3)(2n-1)(2n+1)}+\dots\right),$$

- Hopf algebra techniques can be used to evaluate random graph models.

Bounds

- There are many ways to impose bounds on the value of Feynman integrals Bender and Wu [1969].
- Interesting algebraic structure: The 'Hepp-bound'.
- Renormalization group invariant part of the amplitude is bounded Panzer [2016]:

$$\mathcal{P}(\Gamma) = \int \frac{d\Omega}{\psi^{\frac{D}{2}}} \leq \sum_{\emptyset \subset \gamma_1 \subset \cdots \subset \gamma_{n-1} \subset \Gamma} \frac{1}{\omega_D(\gamma_1) \cdots \omega_D(\gamma_n)}$$

- Sum over all flags, maximal chains of 1PI subdiagrams of Γ.
- ω_D assigns the superficial degree of divergence to the subgraph γ_i.

- These bounds can be summed over all diagrams.
- The generating function for the sum fulfills a non-linear ODE for instance in ϕ^4 MB [2017]:

$$
\begin{aligned}
(\frac{1}{2}x\partial_x - 1)F(x) = \frac{1}{2}\hbar \Bigg(& \partial_x^2 \log \frac{1}{1-F(x)} \\
& - \left[\left(1 + \frac{x^2}{2}\partial_\xi^2\right) \partial_\xi^2 \log \frac{1}{1-F(\xi)} \right]_{\xi=0} \Bigg)
\end{aligned}
$$

- Also carries interesting Hopf-algebraic structures.
- Related to combinatorial constructions on graphs: Ear decompositions and Fulkerson conjecture.

Summary

- Renormalization together with the divergence of the perturbation expansion shows very interesting mathematical structures.
- Hopf algebra techniques enable us to extend the notion of renormalization to evaluate restricted random graph models.
- Similar structures can be used to describe bounds for diagrams, which can be summed easily.
- Hints that we may setup approximations for Feynman integrals that become more accurate the larger the diagram gets.

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