Generating Asymptotics for factorially divergent sequences

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Overview

This work is concerned with sequences a_n, which admit an asymptotic expansion of the form,

$$a_n = \alpha^{n+\beta}\Gamma(n+\beta)\underbrace{\left(c_0 + \frac{c_1}{\alpha(n+\beta-1)} + \frac{c_2}{\alpha^2(n+\beta-1)(n+\beta-2)} + \dots\right)}_{\text{idea: interpret coefficients } c_k \text{ as another power series}},$$

for large n and some $\alpha, \beta \in \mathbb{R}_{>0}$ and $c_k \in \mathbb{R}$.

- Sequences of this type appear in many enumeration problems, which deal with coefficients of factorial growth.
- Interpreted as formal power series, these objects carry a rich algebraic structure. They form a subring of $\mathbb{R}[[x]]$ which is closed under composition and inversion of power series.
- An 'asymptotic derivation' can be defined which maps a power series to its asymptotic expansion.
- Using this formalism, the asymptotic expansions of implicitly defined power series can be obtained in closed form.

Definition

- For given $\alpha, \beta \in \mathbb{R}_{>0}$ let $\mathbb{R}[[x]]^\alpha_\beta$ be the subset of $\mathbb{R}[[x]]$, such that $f \in \mathbb{R}[[x]]^\alpha_\beta$ if and only if there exists a sequence of real numbers $(c_k^f)_{k\in\mathbb{N}_0}$, which fulfills

$$f_n = \sum_{k=0}^{R-1} c_k^f \alpha^{n+\beta-k}\Gamma(n+\beta-k) + \mathcal{O}\left(\alpha^n\Gamma(n+\beta-R)\right) \qquad \forall R \in \mathbb{N}_0. \tag{1}$$

- Let $\mathcal{A}^\alpha_\beta : \mathbb{R}[[x]]^\alpha_\beta \to \mathbb{R}[[x]]$ be the map

$$(\mathcal{A}^\alpha_\beta f)(x) = \sum_{k=0}^{\infty} c_k^f x^k.$$

There is a unique asymptotic expansion $(c_k^f)_{k\in\mathbb{N}_0}$ for every $f \in \mathbb{R}[[x]]^\alpha_\beta$, therefore $\mathcal{A}^\alpha_\beta$ is well-defined.

Statement of results

The subspace $\mathbb{R}[[x]]^\alpha_\beta$ is closed under **addition**, **multiplication**, **composition** and **inversion**.

- $\mathcal{A}^\alpha_\beta$ is a **linear** operator: With $f, g \in \mathbb{R}[[x]]^\alpha_\beta$

$$(\mathcal{A}^\alpha_\beta(f+g))(x) = (\mathcal{A}^\alpha_\beta f)(x) + (\mathcal{A}^\alpha_\beta g)(x). \tag{2}$$

- $\mathcal{A}^\alpha_\beta$ is a **derivation**: With $f, g \in \mathbb{R}[[x]]^\alpha_\beta$ and $(f\cdot g)(x) := f(x)g(x)$

$$(\mathcal{A}^\alpha_\beta(f\cdot g))(x) = f(x)(\mathcal{A}^\alpha_\beta g)(x) + g(x)(\mathcal{A}^\alpha_\beta f)(x). \tag{3}$$

- $\mathcal{A}^\alpha_\beta$ fulfills a **chain rule**: With $f, g \in \mathbb{R}[[x]]^\alpha_\beta$, $g_0 = 0$ and $g_1 = 1$

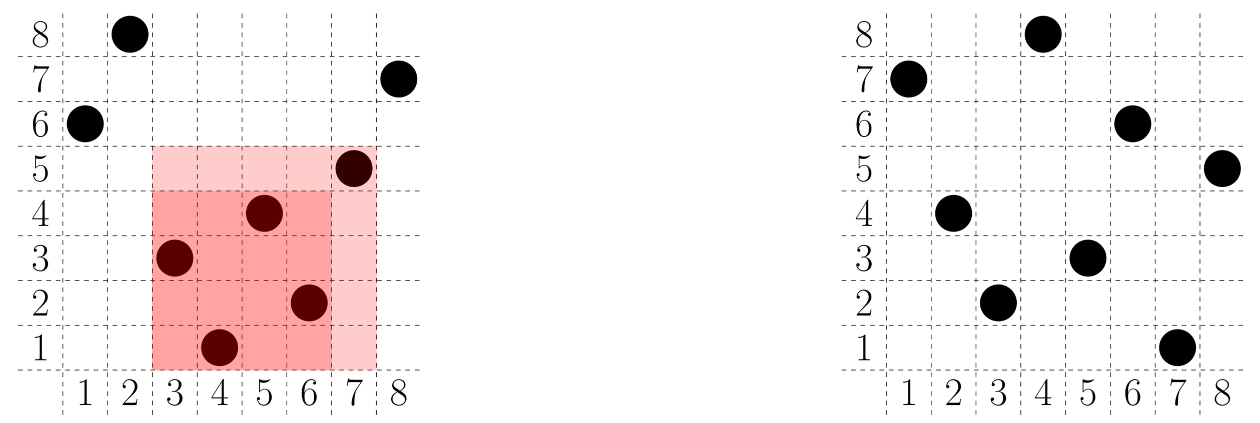
$$(\mathcal{A}^\alpha_\beta(f\circ g))(x) = f'(g(x))(\mathcal{A}^\alpha_\beta g)(x) + \left(\frac{x}{g(x)}\right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}}(\mathcal{A}^\alpha_\beta f)(g(x)). \tag{4}$$

- The compositional **inverse** g^{-1} of a power series $g \in \mathbb{R}[[x]]^\alpha_\beta$ with $g_0 = 0$ and $g_1 = 1$ fulfills

$$(\mathcal{A}^\alpha_\beta g^{-1})(x) = -{g^{-1}}'(x)\left(\frac{x}{g^{-1}(x)}\right)^\beta e^{\frac{g^{-1}(x)-x}{\alpha x g^{-1}(x)}}(\mathcal{A}^\alpha_\beta g)(g^{-1}(x)). \tag{5}$$

These results generalize Bender as well as Bender and Richmond [2, 3].

Application: Simple permutations



a: non-simple permutation **b:** simple permutation

Figure 1: Examples of simple and non-simple permutations. The (non-trivial) intervals that map to intervals are indicated by red squares.

Setup

- A permutation is called simple [1] if it does not map a non-trivial interval to another interval.
- A permutation $\pi \in S_n$ is simple if and only if

$$\pi([i,j]) \neq [k,l] \text{ for all } i,j,k,l \in [1,n] \text{ with } 2 \leq |[i,j]| \leq n-1.$$

- Consider $F(x) = \sum_{n=1}^\infty n!x^n$, the generating function of *all* permutations and
- $S(x) = \sum_{n=4}^\infty S_n x^n$, the generating function of *simple* permutations.
- The generating functions $F(x)$ and $S(x)$ are related [1],

$$\frac{F(x) - F(x)^2}{1+F(x)} = x + S(F(x)). \tag{6}$$

- This can be solved iteratively for the coefficients of $S(x)$:

$$S(x) = 2x^4 + 6x^5 + 46x^6 + 338x^7 + 2926x^8 + \dots$$

Complete asymptotic expansion of S_n

1. We can extract the asymptotics systematically, because

$$F(x) = \sum_{n=1}^\infty n!x^n = \sum_{n=1}^\infty 1^{n+1}\Gamma(n+1)x^n \in \mathbb{R}[[x]]^1_1 \Rightarrow (\mathcal{A}^1_1 F) = 1.$$

2. Apply the $\mathcal{A}^1_1$-derivative to both sides of eq. (6),

$$\mathcal{A}^1_1\left(\frac{F(x)-F(x)^2}{1+F(x)}\right) = \mathcal{A}^1_1\left(x + S(F(x))\right).$$

3. Use the product and chain rules in eqs. (2)-(4) to evaluate the asymptotic derivative,

$$\frac{1-2F(x)-F(x)^2}{(1+F(x))^2}(\mathcal{A}^1_1 F)(x) = S'(F(x))(\mathcal{A}^1_1 F)(x) + \frac{x}{F(x)}e^{\frac{F(x)-x}{xF(x)}}(\mathcal{A}^1_1 S)(F(x)),$$

4. Use eq. (6) and $(\mathcal{A}^1_1 F) = 1$ to simplify the result,

$$(\mathcal{A}^1_1 S)(x) = \frac{1}{1+x}\frac{1-x-(1+x)\frac{S(x)}{x}}{1+(1+x)\frac{S(x)}{x^2}}e^{-\frac{2+(1+x)\frac{S(x)}{x^2}}{1-x-(1+x)\frac{S(x)}{x}}}.$$

This function generates the coefficients of the asymptotic expansion of S_n. It can be expanded,

$$(\mathcal{A}^1_1 S)(x) = e^{-2}\left(1 - 4x + 2x^2 - \frac{40}{3}x^3 - \frac{182}{3}x^4 - \frac{7624}{15}x^5 + \dots\right),$$

and translated into the asymptotic expansion of S_n. By eq. (1),

$$S_n = e^{-2}\left(n! - 4(n-1)! + 2(n-2)! - \frac{40}{3}(n-3)! - \frac{182}{3}(n-4)! - \frac{7624}{15}(n-5)! + \dots\right).$$

Albert, Atkinson and Klazar [1] obtained the first three terms of this expansion using different techniques.

Application: Connected chord diagrams

a: disconnected chord diagram **b:** connected chord diagram

Figure 2: Examples of connected and disconnected chord diagrams. The red rectangles indicate the connected components of the disconnected diagram.

Setup

- A chord diagram with n-chords is a set of $2n$ points, which are labeled by integers $1,\dots,2n$ and connected in disjoint pairs by n-chords. There are $(2n-1)!!$ of such diagrams.
- A chord diagram is *connected* if no set of chords can be separated from the remaining chords by a line which does not cross any chords.
- Let $I(x) = \sum_{n=0}(2n-1)!!x^n$, the generating function of *all* chord diagrams with n chords
- and $C(x) = \sum_{n=0} C_n x^n$, the generating function of *connected* chord diagrams with n chords.
- The power series $I(x)$ and $C(x)$ are related [5],

$$I(x) = 1 + C(xI(x)^2). \tag{7}$$

- This can be solved iteratively for the coefficients of $C(x)$:

$$C(x) = x + x^2 + 4x^3 + 27x^4 + 248x^5 + \dots$$

Complete asymptotic expansion of C_n

1. Recall that $(2n-1)!! = \frac{2^{n+\frac{1}{2}}}{\sqrt{2\pi}}\Gamma(n+\frac{1}{2})$, therefore

$$I(x) = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^\infty 2^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})x^n \in \mathbb{R}[[x]]^2_{\frac{1}{2}} \Rightarrow (\mathcal{A}^2_{\frac{1}{2}} I)(x) = \frac{1}{\sqrt{2\pi}}.$$

2. Analogously to the middle column: Apply $\mathcal{A}^2_{\frac{1}{2}}$ to both sides of eq. (7) and simplify to obtain the generating function of the asymptotic expansion:

$$(\mathcal{A}^2_{\frac{1}{2}} C)(x) = \frac{1}{\sqrt{2\pi}}\frac{x}{C(x)}e^{-\frac{1}{2x}(2C(x)+C(x)^2)}.$$

This generating function can be expanded to obtain the asymptotic expansion of C_n explicitly:

$$C_n = e^{-1}\left((2n-1)!! - \frac{5}{2}(2n-3)!! - \frac{43}{8}(2n-5)!! - \frac{579}{16}(2n-7)!! - \frac{44477}{128}(2n-9)!! + \dots\right).$$

The first coefficient of this expansion was previously calculated by Stein and Everett [6].

Key references

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