Combinatorics of Feynman diagrams and algebraic lattice structure in QFT

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Motivation

Why study the combinatorics of Feynman diagrams?

- (Divergent) perturbation expansions in QFTs dominated by number of diagrams.
- Number of generators of the Hopf/Lie algebra of Feynman diagrams.

Zero dimensional Quantum Field Theory

- What is the simplest way to analyze the combinatorics of Feynman diagrams?
- Zero dimensional QFTs!
- Extensively studied[2].
- Idea: Replace the path integral by an ordinary integration.

[2] Cvitanović, Lautrup, and Pearson 1978.

Zero dimensional Quantum Field Theory

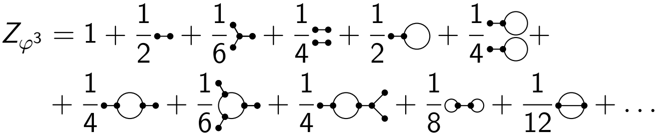
- For φ^k-theory in zero dimensions:

$$Z_{\varphi^k}(j,\lambda) := \int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi}}\, e^{-\frac{\varphi^2}{2}+\lambda\frac{\varphi^k}{k!}+j\varphi},$$

  where λ 'counts' the number of vertices and j the number of external edges.
- This integral is meant to be calculated perturbatively, i.e. by termwise integration:

$$\widetilde{Z}_{\varphi^k}(j,\lambda) := \sum_{n,m\geq 0} \frac{1}{n!m!} \int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi}} \left\{ e^{-\frac{\varphi^2}{2}} \left(\frac{\lambda\varphi^k}{k!}\right)^n (j\varphi)^m \right\}.$$

- Diagrammatically:



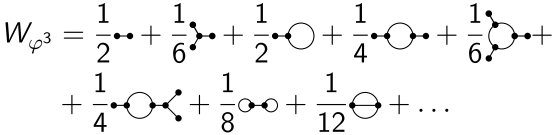
- Analogous for Yukawa, QED, quenched QED, QCD, ...

Zero dimensional Quantum Field Theory

- Use the exponential formula to obtain the connected diagrams:

$$W(j, \lambda) = \log(Z(j, \lambda)).$$

- Diagrammatically:



- Calculate the 'classical' field

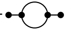
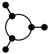
$$\varphi_c(j, \lambda) := \frac{\partial W}{\partial j},$$

- Shift source variable $j \to j' + j_0$ such that $\varphi_c(j')$ vanishes at $j' = 0$.

- Perform a Legendre transformation to obtain the effective action:

$$\Gamma(\varphi_c, \lambda) := W - j'\varphi_c,$$

- Γ is the generating function for all 1PI Feynman diagrams.
- Diagrammatically:

$\Gamma_{\varphi^3} = -\frac{1}{2}$  $+ \frac{1}{6}$  $+ \frac{1}{4}$  $+ \frac{1}{6}$  $+ \frac{1}{12}$  $+ \ldots$

- In the following: Parametrize Γ with $\hbar$ instead of λ to count loops instead of vertices $\rightarrow \Gamma(\varphi_c, \hbar)$.

The Hopf algebra structure of 1PI Feynman diagrams

Starting point for the Hopf algebra of Feynman diagrams:

- $\mathcal{H}^{\text{fg}}$ is the $\mathbb{Q}$-algebra generated by all mutually non-isomorphic 1PI diagrams.
- With Disjoint union as multiplication, a unit u, a counit ϵ
- and the coproduct encapsulating the BPHZ-algorithm:

$$\Delta : \mathcal{T} \to \mathcal{H}^{\text{fg}} \otimes \mathcal{H}^{\text{fg}}$$

$$\Gamma \mapsto \sum_{\substack{\gamma \subset \Gamma \\ \gamma = \prod \gamma_i \\ \text{with each } \gamma_i \text{ 1PI and sup.div.}}} \gamma \otimes \Gamma / \gamma$$

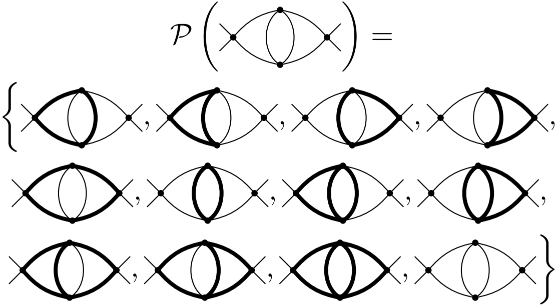
$\mathcal{H}^{\text{fg}}$ becomes a Hopf algebra.

- $\mathcal{H}^{\text{fg}}$ is equipped with a grading given by the loop number.

$$\mathcal{H}^{\text{fg}} = \bigoplus_{L \geq 0} \mathcal{H}^{\text{fg}^{(L)}}$$

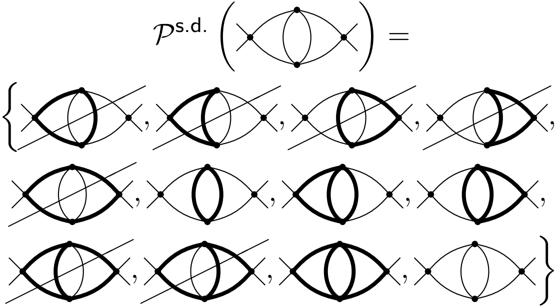
Example

Take all 1PI sub-diagrams of a graph:

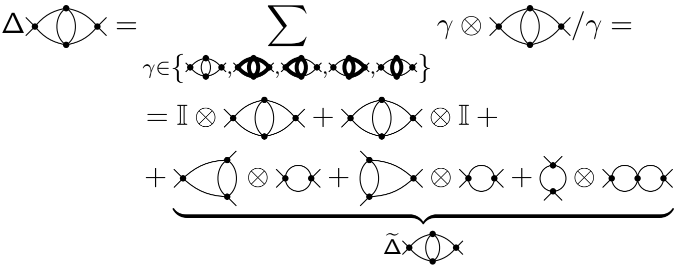


Example

Keep only the superficially divergent ones:



Example



- Tool to calculate finite amplitudes: The group of characters, $G_{\mathcal{A}}^{\mathcal{H}^{\text{fg}}}$.
- Consists of algebra morphisms $\mathcal{H}^{\text{fg}} \to \mathcal{A}$. With $\mathcal{A}$ a unital algebra.
- Product of $\phi, \psi \in G_{\mathcal{A}}^{\mathcal{H}^{\text{fg}}}$ is defined as,

$$\phi * \psi = m_{\mathcal{A}} \circ (\phi \otimes \psi) \circ \Delta.$$

- The unit is $\text{u}_{\mathcal{A}} \circ \epsilon_{\mathcal{H}^{\text{fg}}}$.
- The inverse can be expressed using the antipode S on $\mathcal{H}^{\text{fg}}$, $m \circ (S \otimes \text{id}) \circ \Delta = \text{u} \circ \epsilon$:

$$\phi^{*-1} = \phi \circ S$$

- ϕ denotes the character which maps a Feynman diagram $\in \mathcal{H}^{\text{fg}}$ to its amplitude.
- This amplitude is infinite.
- We are interested in the renormalized amplitude given by,

$$\phi_R := S_R^\phi * \phi.$$

- S_R^ϕ is the 'twisted' antipode defined as

$$S_R^\phi := R \circ \phi \circ S$$

For a multiplicative renormalization scheme.

From diagram counting to the Hopf algebra

- Use the simple Feynman rules of 0-dimensional QFT:

$$\phi : \mathcal{H}^{\text{fg}} \to \mathbb{Q}[[\hbar]], \gamma \mapsto \hbar^{|\gamma|}.$$

For a 1PI diagram γ and $|\gamma|$ its loop number.

- Connect to the path integral formulation, e.g. for φ^3-theory:

$$\frac{\partial^2 \Gamma}{\partial \varphi_c^2} = \phi(X^{-}) \qquad \frac{\partial^3 \Gamma}{\partial \varphi_c^3} = \phi(X^{\succ\!-})$$

where

$$X^{-} := 1 - \sum_{\substack{\gamma \text{ 1PI} \\ \text{res}\,\gamma = -}} \frac{\gamma}{|\operatorname{Aut} \gamma|} \qquad X^{\succ\!-} := 1 + \sum_{\substack{\gamma \text{ 1PI} \\ \text{res}\,\gamma = \succ\!-}} \frac{\gamma}{|\operatorname{Aut} \gamma|}.$$

- Use the toy renormalization scheme: $R = \text{id}$

$$S_R^{\phi} = S^{\phi} = \phi \circ S$$

  This amounts to ‘renormalization of zero dimensional QFT’.

- Using this the counterterms or Z-factors can be obtained,

$$Z^{-} = S^{\phi}(X^{-})$$
$$Z^{\succ\!\!-} = S^{\phi}(X^{\succ\!\!-}).$$

- Explicitly: Using the combinatorial form of Dyson's equation[3]

$$\Delta X^r = \sum_{L \geq 0} Q^{2L} X^r \otimes X^r|_L$$

with the invariant charge $Q := \frac{X^{>\!\!-}}{(X^-)^{\frac{3}{2}}}$.

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$$\Rightarrow Z^r = \frac{1}{\phi(X^r)[\hbar(S^\phi(Q)[\hbar])^2]} \qquad \forall r \in \{>\!\!-, -\}$$

[3]Kreimer 2006.

- The generating function $Z^{\succ\!-}(\hbar)$ counts primitive diagrams[4].
- But why is this the case? Does it work for general QFTs?

$\Rightarrow$ Study S^{ϕ}.

[4] Cvitanović, Lautrup, and Pearson 1978.

The incidence Hopf algebra of posets

- $\mathcal{H}^{\mathrm{P}}$ is the $\mathbb{Q}$-algebra generated by all mutually non-isomorphic partially ordered sets (posets). With a unique smallest element $\hat{0}$ and a unique largest element $\hat{1}$.
- With Cartesian product as multiplication of two posets P_1, P_2:

$$P_1 \cdot P_2 = \{(s,t) : s \in P_1 \text{ and } t \in P_2\}$$
$$\text{with } (s,t) \leq (s',t') \text{ iff } s \leq s' \text{ and } t \leq t'$$

- and the coproduct[5]

$$\Delta : \mathcal{H}^{\mathrm{P}} \to \mathcal{H}^{\mathrm{P}} \otimes \mathcal{H}^{\mathrm{P}}, P \mapsto \sum_{x \in P} [\hat{0}, x] \otimes [x, \hat{1}],$$

where $[x,y]$ is the interval, the subset $\{z \in P : x \leq z \leq y\}$.

[5]Schmitt 1994.

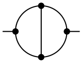
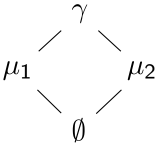
Lemma 1[6]

There is a Hopf algebra morphism,

$$\xi : \mathcal{H}^{\mathsf{fg}} \to \mathcal{H}^{\mathsf{P}}, \gamma \mapsto \mathcal{P}^{\mathsf{s.d.}}(\gamma),$$

mapping a 1PI diagram to its poset of divergent subdiagrams, ordered by inclusion, is a Hopf algebra morphism.

$$\text{For example } \forall \gamma \in \mathrm{Prim}(\mathcal{H}^{\mathsf{fg}}) : \xi(\gamma) = \begin{array}{c} \gamma \\ | \\ \emptyset \end{array},$$

or $\xi\Big($  $\Big) =$ 

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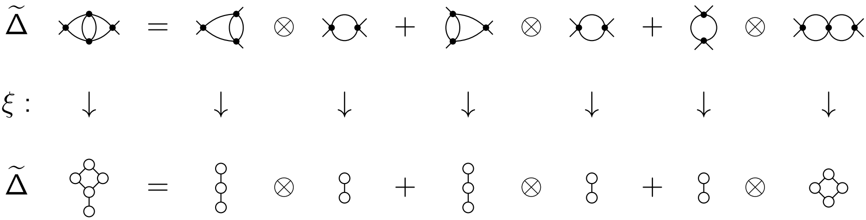
[6] Borinsky (in preparation).

- The map ξ is compatible with the Hopf algebra structure:

$$\xi \circ m_{\mathcal{H}^{\text{fg}}} = m_{\mathcal{H}^{\text{P}}} \circ (\xi \otimes \xi)$$
$$(\xi \otimes \xi) \circ \Delta_{\mathcal{H}^{\text{fg}}} = \Delta_{\mathcal{H}^{\text{P}}} \circ \xi$$

Example:



- The antipode is also compatible:

$$\xi \circ S_{\mathcal{H}^{\text{fg}}} = S_{\mathcal{H}^{\text{P}}} \circ \xi$$

- For a subspace $\mathcal{H}^{\text{fg}^{(L)}} \subset \mathcal{H}^{\text{fg}}$, ξ the toy Feynman rules ϕ act as a characteristic function on $\mathcal{H}^{\text{P}}$:

$$\phi(x) = \hbar^L \phi' \circ \xi(x) \qquad\qquad \forall x \in \mathcal{H}^{\text{fg}^{(L)}}$$

where $\phi' : \mathcal{H}^{\text{P}} \to \mathbb{Q}, P \mapsto 1$.

- Eventually, we want to calculate $S^\phi = \phi \circ S_{\mathcal{H}^{\mathsf{fg}}}$.
- Can be obtained in $\mathcal{H}^{\mathsf{P}}$ for elements in $\mathcal{H}^{\mathsf{fg}^{(L)}} \subset \mathcal{H}^{\mathsf{fg}}$:

$$S^\phi = \hbar^L \phi' \circ \xi \circ S_{\mathcal{H}^{\mathsf{fg}}} = \hbar^L \phi' \circ S_{\mathcal{H}^{\mathsf{P}}} \circ \xi$$

- $\mu := \phi' \circ S_{\mathcal{H}^{\mathsf{P}}}$ is a well studied object, called the möbius function of a poset[7].
- Especially interesting are möbius functions on lattices.

[7] Schmitt 1994; Stanley 1997.

Lattices

- A lattice L is a poset with with a unique greatest and a unique smallest element, $\hat{1}$, $\hat{0}$ and two additional binary operations:
- The join of two elements $x, y \in L$:

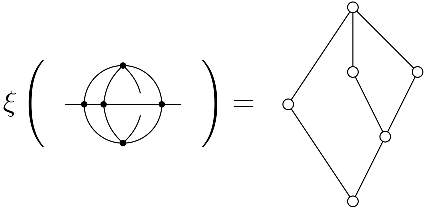
$$x \vee y := \text{ unique smallest element } z, z \geq x \text{ and } z \geq y$$

- and the meet,

$$x \wedge y := \text{ unique greatest element } z, z \leq x \text{ and } z \leq y.$$

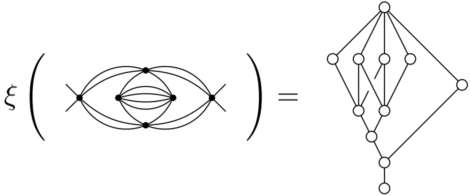
More examples

- In φ^4-theory in 4 dimensions:



This poset is a lattice without a grading.

- In φ^6-theory in 3 dimensions:



This poset is not a lattice.

Theorem 1[8]

In all renormalizable QFTs with vertex valency ≤ 4, ξ maps Feynman diagrams to lattices.

- Join, $\vee$, is defined as the union of two subdiagrams.
- Meet, $\wedge$, is obtained by dualisation.

$\Rightarrow$ Physical QFTs carry a lattice structure.

- Encodes the 'overlapping' structure of the divergences.
- Remark: Diagrams with only logarithmic subdivergences map to distributive lattices[9].

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[8] Borinsky (in preparation).

[9] Berghoff 2014.

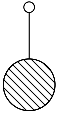
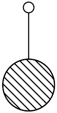
- Why does $S^{\phi}(X$$)$ count primitive diagrams?

Theorem 2[10]

- In a theory with only three-valent vertices, the lattice $\xi(\gamma_v)$ has always one coatom for a vertex diagram γ_v.
- In such a theory, the lattice $\xi(\gamma_p)$ has always two coatoms for a propagator diagram $\gamma_p \neq$ .

[10]Borinsky (in preparation).

- Using Rota's Crosscut Theorem[11]:

$S^{\phi}\Big($  $\Big) = \mu\Big($  $\Big) = 0$ and $S^{\phi}\big($  $\big) = -1$

$$\Rightarrow S^{\phi}(X^{\succ\!\!-}) = 1 - \phi \circ P_{\mathrm{Prim}(\mathcal{H}^{\mathrm{fg}})}(X^{\succ\!\!-})$$

$$S^{\phi}(X^{-}) = 1 + \frac{1}{2}\hbar\left(1 - \phi \circ P_{\mathrm{Prim}(\mathcal{H}^{\mathrm{fg}})}(X^{\succ\!\!-})\right),$$

where $P_{\mathrm{Prim}(\mathcal{H}^{\mathrm{fg}})}$ projects to the primitive elements of $\mathcal{H}^{\mathrm{fg}}$.

[11] Stanley 1997.

Applications

- Back to zero dimensional QFT:

$$Z_{\varphi^3}(j,\hbar) := \int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi\hbar}}\; e^{\frac{1}{\hbar}\left(-\frac{\varphi^2}{2}+\frac{\varphi^3}{3!}+j\varphi\right)}.$$

- Can be 'renormalized' by introducing Z factors and shifting the source $j \to j' + j_0$:

$$Z^{\text{ren}}_{\varphi^3}(j',\hbar) := \int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi\hbar}}\; e^{\frac{1}{\hbar}\left(-Z^{-}(\hbar)\frac{\varphi^2}{2}+Z^{\succ\!-}(\hbar)\frac{\varphi^3}{3!}+j'\varphi+j_0(\hbar)\varphi\right)},$$

- Using a contour integration on a specific order in j',

$$z^{\text{ren}}_{k,n} = \frac{1}{2\pi i}\oint \frac{d\hbar}{\hbar^{1+n}}\frac{\partial^k}{\partial j'^k} Z^{\text{ren}}_{\varphi^3}\Big|_{j'=0},$$

coefficients can be extracted.

- Asymptotically all diagrams are connected and 1PI[12].
- Therefore the probabilities of a random Feynman diagram to be primitive can be obtained.

[12] Wright 1970.

$$z^{\text{ren}}_{k,n} = \frac{1}{2\pi i} \oint \frac{d\hbar}{\hbar^{1+n}} \frac{\partial^k}{\partial j'^k} Z^{\text{ren}}_{\varphi^3}\Big|_{j'=0},$$

- Using the saddle-point expansion the asymptotic behaviour of $z^{\text{ren}}_{k,n}$ can be analyzed. For instance for the  diagrams in φ^3-theory:

$$\lim_{n\to\infty} \frac{z^{\text{ren}}_{3,n}}{z_{3,n}} = e^{-\frac{10}{3}}$$

with $z_{3,n} = \frac{(6n+5)!!}{(2n+1)!(3!)^{2n+1}} = \frac{3!}{2\pi e}\left(\frac{n}{e(3!)^2}\right)^{n+1} + O(n^{-1})$.

- Similar we obtain for

  Yukawa fermion scalar vertex:

$$\lim_{n\to\infty} \frac{z_n^{\text{ren}}}{z_n} = e^{-\frac{7}{2}}$$

  QED fermion photon vertex:

$$\lim_{n\to\infty} \frac{z_n^{\text{ren}}}{z_n} = e^{-\frac{5}{2}}$$

  Quench-QED fermion photon vertex:

$$\lim_{n\to\infty} \frac{z_n^{\text{ren}}}{z_n} = e^{-2}$$

- (Quenched) QED primitives can be enumerated by other methods but asymptotics are more difficult.[13].
- In this case there is a similarity to "irreducible partitions"[14].
- One result in this direction[15], agrees with the asymptotic calculation for Quench QED.
- Connection between primitive diagrams and irreducible partitions?

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[13] Broadhurst 1999; Molinari and Manini 2006.

[14] Beissinger 1985.

[15] Kleitman 1970.

Conclusions

- The lattice structure of Quantum Field Theories is useful to analyze combinatorics of the counterterms.
- I.e. to quantify the divergence stemming from the 'explosion' of diagrams.

$\Rightarrow$ Could be used for estimates for the asymptotic behaviour of Green's functions, β functions, etc.

- Explicit results on the number of primitive elements can be obtained in cases with only 3-valent vertices.